SOLVED EXAMPLES

1) $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = \log x$ ---- (1)

Soln: let us assume $x = e^z$

Then $z = \log x$.

So, $x \frac{dy}{dx} = \frac{dy}{dz}$ and $x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$.

Applying the substitution in (1) we get;

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + 4\frac{dy}{dz} + 2y = z.$$

$$\Rightarrow \frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = z \text{ ----- (2)}$$

The auxillary equation is :- $m^2 + 3m + 2 = 0$

$$\Rightarrow (m^2 + 2m)(+ m + 2) = 0$$

$$\Rightarrow m(m+2) + 1(m+2) = 0$$

$$\Rightarrow (m+1)(m+2) = 0$$

$$\Rightarrow m = -1, -2.$$

So the C.F of (2) is :-

$$y_c = c_1 e^{-z} + c_2 e^{-2z}$$

Now, PI $y_p = \frac{1}{F(D')} F(z)$. where $D' = \frac{d}{dz}$

$$= \frac{1}{(D'^2 + 3D' + 2)} z.$$

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$$\Rightarrow y_p = \frac{1}{2\left(1 + \frac{3D'}{2} + \frac{D'^2}{2}\right)} z$$

$$\Rightarrow y_p = \frac{1}{2}\left(1 + \left(\frac{3D'}{2} + \frac{D'^2}{2}\right)\right)^{-1} z.$$

$$= \frac{1}{2}\left[1 - \frac{3D'}{2}\right] z$$

[Higher order terms vanish as degree of z is 1 so it will be differentiated once only].

$$= \frac{1}{2}\left(z - \frac{3}{2}\right)$$

$$= \frac{1}{2}z - \frac{3}{4}$$

So, $y_p = \frac{1}{2}z - \frac{3}{4}$.

The general solution of (2) is:

$$y = y_c + y_p = c_1 e^{-z} + c_2 e^{-2z} + \frac{1}{2}z - \frac{3}{4}.$$

Resubstituting $z = \log x$ we get;

The general solution of (1) is:

$$y = c_1 x^{-1} + c_2 x^{-2} + \frac{1}{2}\log x - \frac{3}{4}.$$

$$\Rightarrow y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{1}{2}\log x - \frac{3}{4}.$$

2) $(x^2 D^2 + xD - 1)y = \sin(\log x) + x\cos(\log x)$. ---- (1)

Soln :- Let $x = e^z$, then $z = \log x$.

$xD = x\frac{dy}{dx} = \frac{dy}{dz}$ and $x^2 D^2 = x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$.

Using this substitution in (1) we get;

$\frac{d^2y}{dz^2} - \cancel{\frac{dy}{dz}} + \cancel{\frac{dy}{dz}} - 1 = \sin z + e^z \cos z$.

$\frac{d^2y}{dz^2} - 1 = \sin z + e^z \cos z$ ---- (2)

The auxillary equation of (2) is :-

$m^2 - 1 = 0$

$\Rightarrow m^2 = 1$

$\Rightarrow m = \pm 1$

So the C.F of (2) is :-

$y_c = c_1 e^{-z} + c_2 e^z$.

The PI $y_p = \frac{1}{F(D')} F(z)$.

$= \frac{1}{(D'^2 - 1)} \sin z + e^z \cos z$.

$= \frac{1}{D'^2 - 1} \sin z + \frac{1}{D'^2 - 1} e^z \cos z$.

$$\rightarrow \quad y_p = \frac{1}{i^2 - 1} \sin z + e^z \frac{1}{(D'+1)^2 - 1} \cos z$$

$$= -\frac{1}{2} \sin z + e^z \frac{1}{D'^2 + 2D'} \cos z .$$

$$= -\frac{1}{2} \sin z + e^z \frac{1}{-1 + 2D'} \cos z .$$

$$= -\frac{1}{2} \sin z + e^z \frac{2D'+1}{(2D'-1)(2D'+1)} \cos z .$$

$$= -\frac{1}{2} \sin z + e^z \frac{2D'+1}{4D'^2 - 1} \cos z .$$

$$= -\frac{1}{2} \sin z + e^z \frac{2D'+1}{-4-1} \cos z .$$

$$= -\frac{1}{2} \sin z - \frac{1}{5} e^z (2D'+1) \cos z .$$

$$= -\frac{1}{2} \sin z - \frac{1}{5} e^z (\cos z - 2\sin z) .$$

$$= -\frac{1}{2} \sin z + \frac{1}{5} e^z (2\sin z - \cos z) .$$

So the general solution of (2) is :-

$$y = y_c + y_p = c_1 e^{-z} + c_2 e^z - \frac{1}{2} \sin z + \frac{1}{5} e^z (2\sin z - \cos z) .$$

Putting $z = \log x$ above the general soln of (1) is :-

$$y = \frac{c_1}{x} + c_2 x - \frac{1}{2} \sin(\log x) + \frac{x}{5} \left(2\sin(\log x) - \cos(\log x) \right).$$

—— Answer

3) $$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2 \quad \text{---- (1)}$$

Let $x = e^z$ then $z = \log x$

So, $x\frac{dy}{dx} = \frac{dy}{dz}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$.

Using this substitution in (1) we get;

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} - 3\frac{dy}{dz} + 4 = 2\cancel{x}\, e^{2z}$$

$$\Rightarrow \frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 4 = 2e^{2z} \quad \text{---- (2)}$$

The auxillary equation of (2) is :-

$$m^2 - 4m + 4 = 0$$

$$\Rightarrow (m-2)^2 = 0$$

$$\Rightarrow m = 2, 2.$$

So the C.F of (2) is :-

$$y_c = \cancel{c_1 e} \; (c_1 + c_2 z)\, e^{2z}.$$

The P.I $$y_p = \frac{1}{F(D')} F(z)$$

$$= \frac{1}{D'^2 - 4D' + 4} 2e^{2z}.$$

Substituting $D' = 2$ we get the denominator zero.

So,
$$y_p = 2z \frac{1}{2D' - 4} e^{2z}.$$

Substituting $D' = 2$, the denominator is again zero.

So,
$$y_p = \cancel{2}z^2 \frac{1}{\cancel{2}} e^{2z} = z^2 e^{2z}$$

So the general solution of (2) is :-

$$y = y_c + y_p = (c_1 + c_2 z) e^{2z} + z^2 e^{2z}$$

Substituting $z = \log x$ the general solution of (1) is :-

$$y = (c_1 + c_2 \log x) x^2 + x^2 (\log x)^2$$

— Answer